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# STABILITY OF CONDUCTIVITIES IN AN INVERSE PROBLEM IN THE REACTION-DIFFUSION SYSTEM IN ELECTROCARDIOLOGY

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**ABSTRACT.** In this paper, we study the stability result for the conductivities diffusion coefficients to a strongly reaction-diffusion system modeling electrical activity in the heart. To study the problem, we establish a Carleman estimate for our system. The proof is based on the combination of a Carleman estimate and certain weight energy estimates for parabolic systems.

**1. Introduction.** Let  $\Omega \subset \mathbb{R}^N (N \geq 1)$  be a bounded connected open set whose boundary  $\partial\Omega$  is regular enough. Let  $T > 0$  and  $\omega$  be a small nonempty subset of  $\Omega$ . We will denote  $(0, T) \times \Omega$  by  $Q_T$  and  $(0, T) \times \partial\Omega$  by  $\Sigma_T$ .

To state the model of the cardiac electric activity in  $\Omega$  ( $\Omega \subset \mathbb{R}^3$  being the natural domain of the heart), we set  $u_i = u_i(t, x)$  and  $u_e = u_e(t, x)$  to represent the spacial cellular and location  $x \in \Omega$  of the intracellular and extracellular electric potentials respectively. Their difference  $v = u_i - u_e$  is the transmembrane potential. The anisotropic properties of the two media are modeled by intracellular and extracellular conductivity tensors  $M_i(x)$  and  $M_e(x)$ . The surface capacitance of the membrane is represented by the constant  $c_m > 0$ . The transmembrane ionic current is represented by a nonlinear function  $h(v)$ .

The equations governing the cardiac electric activity are given by the coupled reaction-diffusion system:

$$\begin{cases} c_m \partial_t v - \operatorname{div}(M_i(x) \nabla u_i) + h(v) = f \chi_\omega, & \text{in } Q_T, \\ c_m \partial_t v + \operatorname{div}(M_e(x) \nabla u_e) + h(v) = g \chi_\omega, & \text{in } Q_T, \end{cases} \quad (1)$$

where  $f$  and  $g$  are stimulation currents applied to  $\Omega$ . We complete this model with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$u_i = 0, \quad u_e = 0, \quad \text{on } \Sigma_T, \quad (2)$$

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and with initial data for the transmembrane potential

$$v(0, x) = v_0(x), \quad x \in \Omega. \quad (3)$$

It is important to point out that realistic models describing electrical activities include a system of ODEs for computing the ionic current as a function of the transmembrane potential and a series of additional “gating variables”, which aim to model the ionic transfer across the cell membrane.

Assume that the intra and extracellular stimulations are equal:  $f\chi_\omega = g\chi_\omega$ . If  $M_i = \mu M_e$  for some constant  $\mu \in \mathbb{R}$ , then by multiplying the second equation in (1) by  $\mu$  and adding it to the first equation in (1) one gets the first equation in the following parabolic-elliptic system:

$$\begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v) = -h(v) + f\chi_\omega, & \text{in } Q_T, \\ \operatorname{div}(M(x) \nabla u_e) = \operatorname{div}(M_i(x) \nabla v), & \text{in } Q_T, \\ v(0, x) = v_0(x), \quad u_e(0, x) = u_{e,0}(x), & \text{in } \Omega, \\ v = 0, \quad u_e = 0, & \text{on } \Sigma_T. \end{cases} \quad (4)$$

The second equation is obtained by computing the difference of the two equation in (1). Here  $M = M_i + M_e$ . System (4) is known as the monodomain model.

We approximate the above model (4) by the following family of parabolic equations

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -h(v^\varepsilon) + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(0, x) = v_0(x), \quad u_e^\varepsilon(0, x) = u_{e,0}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (5)$$

$\varepsilon$  is a fixed small constant. Since  $v = u_i - u_e$  in the bidomain model, it is natural decompose the initial condition  $v_0$  as  $v_0 = u_{i,0} - u_{e,0}$ . Note that when  $\varepsilon \rightarrow 0$  in (5), we obtain the classical monodomain model.

In this work, we study the stability result for the conductivities diffusion coefficients to the following linearized system of (5) with semi-initial conditions

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -a(t, x) v^\varepsilon + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(\theta, x) = v_\theta(x), \quad u_e^\varepsilon(\theta, x) = u_{e,\theta}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (6)$$

where  $a(t, x)$  and its derivative with respect to  $t$  exists and are bounded in  $Q_T$ . For some  $\theta \in (0, T)$ , the semi-initial conditions  $v_\theta(x)$ ,  $u_{e,\theta}(x)$  are sufficiently regular. The unknown conductivity tensors  $M$  and  $M_e$  are assumed to be sufficiently smooth and shall be kept independent of time  $t$ .

The existence of weak solutions of (1) is proved in [10] by the theory of evolution variational inequalities in Hilbert space. Then Bendahmane and Karlsen [2] proved the existence and uniqueness for a nonlinear version of the bidomain equations (1) by a uniformly parabolic regularization of the system and the Faedo-Galerkin method. Moreover, Bendahmane and Chaves-Silva [1] studied exact null controllability to (1) for each  $\varepsilon > 0$  by establishing estimates for its dual system. To learn more about the cardiac problems, one can refer to the work of Bendahmane et al. [3, 4]. However, it is noted that there is no stability results for the inverse bidomain model.

Since the pioneer work due to A.L. Bukhgeim and M.V. Klibanov [6, 7, 8], who generalized the method of global Carleman estimates in the context of inverse problems, three fundamental issues have been successfully studied: uniqueness, stability in determining coefficients, and numerical methods [16], [13, 14, 17, 20, 23, 24, 15]).

The paper by Cristofol et al. [11] obtains the stability results for reaction-diffusion system of two equations with constant coefficients using a Carleman estimate. Then Sakthivel et al. [21] established the stability results for Lotka-Volterra competition-diffusion system of three equations with variable diffusion coefficients. Our inverse stability results are new because system (6) contains a strong coupling term. The technics we shall discuss are similar to the framework using Carleman estimates for inverse problems but the obtained estimates differs from those of [24], [21] because of the strongly coupled terms.

Let  $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$  be a solution of system (6) with conductivity tensors  $(\tilde{M}_e, \tilde{M})$  and semi-initial data  $(\tilde{v}_\theta^\varepsilon, \tilde{u}_{e,\theta}^\varepsilon)$ . Then setting  $A_1 = v^\varepsilon - \tilde{v}^\varepsilon$ ,  $A_2 = u_e^\varepsilon - \tilde{u}_e^\varepsilon$ ,  $g_1 = M_e - \tilde{M}_e$  and  $g_2 = M - \tilde{M}$ , we obtain

$$\begin{cases} c_m \partial_t A_1 - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla A_1(t, x)) = -a(t, x) A_1(t, x) + F(g_1, \nabla \tilde{v}^\varepsilon), & \text{in } Q, \\ \varepsilon \partial_t A_2 - \operatorname{div}(M(x) \nabla A_2) = \operatorname{div}(M_i(x) \nabla A_1) + G(g_2, \nabla u_e^\varepsilon), & \text{in } Q, \\ A_1(\theta, x) = A_1^\theta(x), \quad A_2(\theta, x) = A_2^\theta(x), & \text{in } \Omega, \\ A_1(t, x) = 0, \quad A_2(t, x) = 0, & \text{on } \Sigma, \end{cases} \quad (7)$$

where

$$F = \frac{\mu}{\mu+1} \operatorname{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$$

and

$$G = \operatorname{div}(g_2(x) \nabla \tilde{u}_e^\varepsilon).$$

Throughout the paper, we make the following assumptions:

**Assumption 1.1.** *The conductivity tensors  $M_e(x)$ ,  $M_i(x)$  and  $M(x)$  are  $C^\infty$ , bounded, symmetric, semi-definite, and elliptic matrixes (there exists  $\beta > 0$  such that  $\Sigma_{i,j}^3 M_{i,j} \xi_i \xi_j \geq \beta |\xi|^2$  for all  $\xi \in \mathbb{R}^3$ ). All their derivatives up to the third order are respectively bounded by the positive constants  $\gamma_1, \gamma_2, \gamma_3$ .*

**Assumption 1.2.** *Assume the bounded measurements  $\partial_t A_1$  and  $\partial_t A_2$  in  $(0, T) \times \omega$  are given. Also  $A_i(\theta, x)$ ,  $\nabla A_i(\theta, x)$ ,  $\Delta A_i(\theta, x)$  and  $\nabla(\Delta A_i(\theta, x))$  for some fixed  $\theta \in (0, T)$ , where  $i = 1, 2$  in  $\Omega$  are given.*

Now the question of interest is whether we can determine the conductivity tensors  $M_e$  and  $M$  by the two measurements.

In details, let  $(v^\varepsilon, u_e^\varepsilon)$  and  $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$  be the solutions of the system (6) with two different conductivities. There exist a constant  $C$  with  $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$ , such that the following estimate holds:

$$\begin{aligned} & \int_{\Omega} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx \\ & \leq C \left( \int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\ & \quad + C \int_{\tilde{\omega}} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx. \end{aligned} \quad (8)$$

**2. A Carleman type estimate.** In this section, we prove the Carleman estimate based on the standard technique for general parabolic equations. In order to frame a Carleman type estimate, we shall first introduce a particular type of weight functions.

**2.1. Weight functions.** First, we introduce weight functions for the parabolic equations given in [12].

Let  $\tilde{\omega} \subset \subset \omega$  be a nonempty bounded set of  $\Omega$ , and  $\psi \in C^2(\bar{\Omega})$  such that

$$\psi(x) > 0, \text{ for any } x \in \Omega,$$

$$\psi(x) = 0, \text{ for any } x \in \partial\Omega,$$

$$|\nabla\psi(x)| > 0, \text{ for any } x \in \bar{\Omega} \setminus \tilde{\omega}.$$

Then we introduce another two weight functions:

$$\phi(t, x) = \frac{e^{\lambda\psi(x)}}{\beta(t)}, \quad (9)$$

$$\alpha(t, x) = \frac{e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi(x)}}{\beta(t)}, \quad (10)$$

where  $\lambda > 1$ ,  $t \in (0, T)$  and  $\beta(t) = t(T - t)$ . Note that the weight function  $\alpha$  is positive, and blows up to  $\infty$  as  $t = 0$  or  $t = T$ . As a result,  $e^{-2s\alpha}$  and  $\phi e^{-2s\alpha}$  are smooth. Even they vanish when  $t = 0$  or  $t = T$ . It can be seen that  $\phi(t, x) \geq C > 0$  for all  $(t, x) \in Q$ , and  $e^{-\epsilon\alpha}\phi^m \leq C < \infty$  for all  $\epsilon > 0$  and  $m \in \mathbb{R}$ .

Before proving the main estimate, we give the following estimates for the two weight functions  $\alpha$  and  $\phi$ . Note that throughout the paper we will denote  $C$  as a generic positive constant. After some computations, we can obtain the following estimates:

$$\begin{cases} |\phi_t| = \frac{|2t-T|}{e^{\lambda\psi}} \phi^2 \leq CT\phi^2, \\ |\alpha_t| = \frac{|2t-T|}{\beta^2} (e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi}) \leq CT\phi^2, \\ |\alpha_{tt}| = \frac{2|T^2-3tT+3t^2|}{\beta^3} (e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi}) \leq CT\phi^3. \end{cases} \quad (11)$$

Furthermore, we also have

$$\begin{cases} \nabla\phi = \lambda\phi\nabla\psi, \\ \nabla\alpha = -\lambda\phi\nabla\psi, \\ \phi^{-1} \leq (\frac{T}{2})^2. \end{cases} \quad (12)$$

Refer to [12] for the details.

**2.2. Main proof of a Carleman type estimate.** Let us set  $Q_\omega = (0, T) \times \omega$ . For each positive integer  $m$ , we denote the Sobolev space of functions in  $L^p(\Omega)$  whose weak derivatives of order less than or equal to  $m$  are also in  $L^p(\Omega)$  with the norm denoted  $\|\cdot\|_{L^p(\Omega)}$ , by  $W^{m,p}(\Omega)$  with  $p > 1$  or  $p = \infty$ . When  $p = 2$ , we denote  $W^{m,p}$  by  $H^m(\Omega)$ . Moreover, let  $L^2(0, T; H^1(\Omega))$  be the space of all equivalent classes of square integrable functions from  $(0, T)$  to  $H^1(\Omega)$ . For the space  $L^2(0, T; L^\infty(\Omega))$ , we define it in the same way.

Let  $A_1$  be the solution of the first equation of (7) with help of using Assumption 1.1. We apply the Carleman estimate (see Theorem 6.1 in [1].) derived for the parabolic equations to the first equation in (7). For  $\lambda > \lambda_0 \geq 1$ ,  $s \leq s_0(T + T^2 + T^4)$ , there exists a constant  $C$  depending on  $\Omega$ ,  $\omega$ ,  $\psi$  and  $\beta$  so that

$$\mathcal{I}(A_1) \leq C \left( \int_Q e^{-2s\alpha} |F|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right), \quad (13)$$

where  $\tilde{\omega} \subset \subset \omega_1 \subset \subset \omega$ , and

$$\begin{aligned} \mathcal{I}(A_1) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_1| + |\Delta A_1|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_1|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_1|^2 dt dx. \end{aligned} \quad (14)$$

Similarly, for  $\lambda > \lambda_0 \geq 1$ ,  $s \geq s_0(T + T^2 + T^4)$ , there exists a constant  $C$  depending on  $\Omega$ ,  $\omega$ ,  $\psi$  and  $\beta$  satisfying

$$\begin{aligned} \mathcal{I}(A_2) &\leq C \left( \int_Q e^{-2s\alpha} (|G|^2 |\nabla(M_i \nabla A_1)|^2) dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\leq C \left( \int_Q e^{-2s\alpha} |G|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx + \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \right) \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx, \end{aligned} \quad (15)$$

with

$$\begin{aligned} \mathcal{I}(A_2) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_2| + |\Delta A_2|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_2|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_2|^2 dt dx. \end{aligned} \quad (16)$$

Now coupling the above inequalities (13) and (15), we have

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq C \left( \int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right) + C \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx \end{aligned}$$

for sufficiently large  $s \geq s_0(T + T^2 + T^4)$  and  $\lambda \geq \lambda_0$ . From the definition of  $\mathcal{I}_1$ , also  $M_i$  and  $\nabla M_i$  being bounded, we obtain

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left( \int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right). \end{aligned} \quad (17)$$

Then it can be summarized as our desired Carleman estimate as follows.

**Theorem 2.1.** *Let  $\psi(x)$ ,  $\phi(t, x)$  and  $\alpha(t, x)$  be defined as in the above subsection,  $a(t, x)$  is a bounded function. Moreover, Assumption 1.1 holds. Then there exist  $\lambda_0$  and  $s_0$  such that for all  $\lambda > \lambda_0 \geq 1$  and sufficiently large enough  $s > s_0$ , the following inequality is true.*

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left( \int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right), \end{aligned}$$

where  $\tilde{C} > 0$  is a constant depending on  $\Omega, T, \omega, \gamma_2$ .

**3. Stability of the conductivities.** In this section, we study the stability of the conductivity tensors  $M_e$  and  $M$ . Then an inequality is established which estimates  $g_1, g_2, \nabla g_1, \nabla g_2$  with an upper bound given by some Sobolev norms of the derivative of  $A_1$  and  $A_2$  over  $Q_\omega$ , certain spatial derivative of  $A_j(\theta, \cdot)$ ,  $j = 1, 2$ , where  $\theta \in (0, T)$  makes  $\frac{1}{\beta(t)}$  attain its minimum value and the Sobolev norm of  $g_1, g_2, \nabla g_1, \nabla g_2$  in a small space  $\tilde{\omega}$ .

First, we let  $B_1 = \partial_t A_1, B_2 = \partial_t A_2$ . Using this and (7), we get the following system:

$$\begin{cases} c_m \partial_t B_1 - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla B_1(t, x)) = -\partial_t a(t, x) A_1(t, x) - a(t, x) B_1 \\ \quad + F'(g_1, \nabla \tilde{v}^\varepsilon), \text{ in } Q_T, \\ \varepsilon \partial_t B_2 - \operatorname{div}(M(x) \nabla B_2) = \operatorname{div}(M_i(x) \nabla B_1) + G'(g_2, \nabla u_e^\varepsilon), \text{ in } Q_T, \\ c_m B_1(\theta, x) = H_1^\theta(x), \quad B_2(\theta, x) = H_2^\theta(x), \text{ in } \Omega, \\ B_1(t, x) = 0, \quad B_2(t, x) = 0, \text{ on } \Sigma_T, \end{cases} \quad (18)$$

where

$$F' = \frac{\mu}{\mu+1} \operatorname{div}(g_1(x) \nabla(\partial_t \tilde{v}^\varepsilon)), \quad G' = \operatorname{div}(g_2(x) \nabla(\partial_t \tilde{u}_e^\varepsilon))$$

and

$$\begin{cases} c_m B_1(\theta, x) = \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla A_1(\theta, x)) - a(\theta, x) A_1(\theta, x) + F|_{t=\theta} = H_1^\theta, \text{ in } Q_T, \\ \varepsilon B_2(\theta, x) = \operatorname{div}(M(x) \nabla A_2(\theta, x)) + \operatorname{div}(M_i(x) \nabla A_1(\theta, x)) + G|_{t=\theta} = H_2^\theta, \text{ in } Q_T, \\ A_1(t, x) = A_1(0, x) + \int_0^t B_1(s, x) ds, \text{ in } Q_T. \end{cases} \quad (19)$$

Indeed, to prove the main result here we need to impose some regularity properties as follows.

**Assumption 3.1.** Suppose  $v_\theta^\varepsilon$  and  $u_{e,\theta}^\varepsilon$  are  $C^3$  real valued functions. Then all their derivatives up to order three are bounded and satisfy  $|\nabla \psi \cdot \nabla v_\theta^\varepsilon| \geq \delta > 0$ ,  $|\nabla \psi \cdot \nabla u_{e,\theta}^\varepsilon| \geq \delta > 0$ , on  $\overline{\Omega} \setminus \tilde{\omega}$ , where  $\tilde{\omega} \subset \subset \omega \subset \subset \Omega$ .

**Assumption 3.2.** Suppose  $(|\Delta \tilde{v}^\varepsilon|, |\Delta \tilde{u}_e^\varepsilon|), (|\nabla(\Delta \tilde{v}^\varepsilon)|, |\nabla(\Delta \tilde{u}_e^\varepsilon)|), (|\nabla(\partial_t \tilde{v}^\varepsilon)|, |\nabla(\partial_t \tilde{u}_e^\varepsilon)|)$  and  $(|\Delta(\partial_t \tilde{v}^\varepsilon)|, |\Delta(\partial_t \tilde{u}_e^\varepsilon)|)$  are bounded by a positive constant.

Before start proving our main conclusion, we need to give the following Lemma 3.3, which will be useful in the following part. We define the following operators  $P_0$  and  $Q_0$  and the initial conditions on  $\alpha$  and  $\phi$  at  $t = \theta$ :

$$P_0 h = \nabla U_\theta \cdot \nabla h, \quad Q_0(e^{-s\alpha^\theta} h) = e^{-s\alpha^\theta} P_0 h \text{ and } \varsigma(\theta, x) = \varsigma^\theta \text{ for } \varsigma = \alpha, \phi.$$

**Lemma 3.3.** Consider the first order partial differential operator  $P_0 h = \nabla U_\theta \cdot \nabla h$ , where  $U_\theta$  satisfies Assumption 3.1. Then there exists a constant  $C > 0$ , such that for sufficiently large enough  $\lambda$  and  $s$ , the following result holds:

$$s^2 \lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \leq C \left( \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 h|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \right),$$

with  $\theta \in (0, T)$  and  $h \in H_0^1(\Omega)$ .

*Proof.* Let  $B_1 = e^{-s\alpha^\theta} h$ , we have

$$Q_0 B_1 = e^{-s\alpha^\theta} P_0(e^{s\alpha^\theta} B_1) = P_0 B_1 + s B_1 P_0 \alpha^\theta, \quad (20)$$

$h \in H_0^1(\Omega)$ . Then we take the square of both sides in (20), multiply  $\frac{1}{\phi^\theta}$  and integrate by parts with respect to space variable for both sides of (20) as follows:

$$\begin{aligned}
& \int_{\Omega} \frac{1}{\phi^\theta} (Q_0 B_1)^2 dx \\
&= \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^\theta} (P_0 \alpha^\theta)^2 dx \\
&\quad + \int_{\Omega} 2s \frac{1}{\phi^\theta} B_1 (P_0 B_1) (P_0 \alpha^\theta) dx \\
&= \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^\theta} \lambda^2 (\phi^\theta)^2 (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
&\quad - \int_{\Omega} 2\lambda s \frac{1}{\phi^\theta} B_1 (\nabla U^\theta \cdot \nabla B_1) (\nabla U^\theta \cdot \phi^\theta \nabla \psi^\theta) dx \\
&= \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
&\quad - \int_{\Omega} 2\lambda s B_1 (\nabla U^\theta \cdot \nabla B_1) (\nabla U^\theta \cdot \nabla \psi^\theta) dx \\
&= \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
&\quad - \int_{\Omega} \lambda s P_0 \psi^\theta \nabla U^\theta \nabla (B_1^2) dx \\
&= \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
&\quad + \int_{\Omega} \lambda s \nabla (P_0 \psi^\theta \nabla U^\theta) B_1^2 dx \\
&\geq \int_{\Omega} s^2 \lambda^2 B_1^2 \phi^\theta |\nabla U^\theta \cdot \nabla \psi^\theta|^2 dx + \int_{\Omega} s \lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx \\
&\geq s^2 \lambda^2 \delta^2 \left( \int_{\Omega} |B_1|^2 \phi^\theta dx - \int_{\tilde{\omega}} |B_1|^2 \phi^\theta dx \right) + \int_{\Omega} s \lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx,
\end{aligned}$$

where we used Assumption 3.1 in the last step. Thus we obtain

$$\begin{aligned}
& s^2 \lambda^2 \delta^2 \left( \int_{\Omega} |B_1|^2 \phi^\theta dx - \int_{\tilde{\omega}} |B_1|^2 \phi^\theta dx \right) \\
& \leq \int_{\Omega} \frac{1}{\phi^\theta} |Q_0 B_1|^2 dx + \int_{\Omega} s \lambda |\nabla (P_0 \psi^\theta \nabla U^\theta)| |B_1|^2 dx.
\end{aligned}$$

From Assumption 3.2 and (12), we have

$$\begin{aligned}
& s^2 \lambda^2 \delta^2 \int_{\Omega} e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
& \leq \int_{\tilde{\omega}} s^2 \lambda^2 \delta^2 e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx + C_1 T^2 \int_{\Omega} s \lambda e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
& \quad + \int_{\Omega} e^{-2s\alpha^\theta} |P_0 h|^2 \frac{1}{\phi^\theta} dx. \tag{21}
\end{aligned}$$

Taking  $\lambda \geq 1$  and  $s \geq \frac{2C_1 T^2}{\delta^2}$ , we conclude the proof.  $\square$

With the help of the Lemma 3.3, we are proving the following proposition.



**Proposition 1.** *Let  $(A_1, A_2)$  be the solution of (7), and  $(B_1, B_2)$  be the solution of (18). Suppose all the conditions of Theorem 2.1 and Assumption 3.1 hold. Then there exists a constant  $C = C(\gamma_1, \gamma_2, \delta) > 0$  such that for sufficiently large enough  $s$  and  $\lambda$  the following estimate is true.*

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\ & \leq C \sum_{j=1}^9 E_j + C s^2 \lambda^2 \int_{\tilde{\omega}} e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx, \end{aligned} \quad (22)$$

for any  $g_1, g_2 \in H_0^2(\Omega)$ , where the functions  $E_j$ , are given as follows:

$$\begin{aligned} E_1 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_1^\theta|^2 dx, \\ E_2 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_2^\theta|^2 dx, \\ E_3 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_1^\theta|^2 dx, \\ E_4 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_2^\theta|^2 dx, \\ E_5 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|A_1^\theta|^2 + |\nabla A_1^\theta|^2 + |\Delta A_1^\theta|^2) dx, \\ E_6 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|\nabla A_2^\theta|^2 + |\Delta A_2^\theta|^2) dx, \\ E_7 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \left( |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) \right. \\ & \quad \left. + |\nabla(g_1 \Delta \tilde{v}_\theta^\varepsilon)|^2 + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon)|^2 \right) dx, \\ E_8 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_1 \Delta \tilde{v}_\theta^\varepsilon|^2 dx, \\ E_9 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon|^2 dx. \end{aligned}$$

*Proof.* Due to the value of the solutions satisfying the first equation in (18) at  $t = \theta$ , and  $F = \text{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$ , from (19) we obtain

$$P_0 g_1 = \nabla \tilde{v}_\theta^\varepsilon \cdot \nabla g_1 = c_m B_1^\theta + a(\theta, x) A_1^\theta - \frac{\mu}{\mu + 1} \text{div}(M_e \nabla A_1^\theta) - g_1 \Delta \tilde{v}_\theta^\varepsilon.$$

Note that we replace  $h$  by  $g_1$  when choosing  $U_\theta$  as  $\tilde{v}_\theta^\varepsilon$ . Therefore, inspired by Lemma 3.3, we get

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \\ & \leq C \left( \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 g_1|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \right) \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |B_1^{\theta}|^2 + |A_1^{\theta}|^2 + \left( \frac{\mu}{\mu+1} \right)^2 (|\nabla M_e|^2 |\nabla A_1^{\theta}|^2 \right. \right. \\
&\quad \left. \left. + |M_e|^2 |\Delta A_1^{\theta}|^2) + |g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}|^2 \right) dx + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_1|^2 dx \\
&\leq C(\gamma_1)(E_1 + E_5 + E_8) + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_1|^2 dx. \tag{23}
\end{aligned}$$

Similarly, from the value of the solutions satisfying the second equation in (18) at  $t = \theta$ , and  $G = \operatorname{div}(g_2(x) \nabla \tilde{u}_{\varepsilon}^{\varepsilon})$ , we obtain

$$P_0 g_2 = \nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla g_2 = \varepsilon B_2^{\theta} - \operatorname{div}(M \nabla A_2^{\theta}) - \operatorname{div}(M_i \nabla A_1^{\theta}) - g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}.$$

It leads to

$$\begin{aligned}
&s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \\
&\leq C \left( \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} |P_0 g_2|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \right) \\
&\leq C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |B_2^{\theta}|^2 + |\nabla M|^2 |\nabla A_2^{\theta}|^2 + |M|^2 |\Delta A_2^{\theta}|^2 \right. \\
&\quad \left. + |\nabla M_i|^2 |\nabla A_1^{\theta}|^2 + |M_i|^2 |\Delta A_1^{\theta}|^2 + |g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}|^2 \right) dx \\
&\quad + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \\
&\leq C(\gamma_2, \gamma_3)(E_2 + E_5 + E_6 + E_9) + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx. \tag{24}
\end{aligned}$$

On the other hand, from the expression of  $P_0 g_1$ , we can see that,

$$\begin{aligned}
P_0 \nabla g_1 &= \nabla \tilde{v}_{\theta}^{\varepsilon} \cdot \nabla (\nabla g_1) = \nabla (\nabla \tilde{v}_{\theta}^{\varepsilon} \cdot \nabla g_1) - \nabla g_1 \Delta \tilde{v}_{\theta}^{\varepsilon} \\
&= c_m \nabla B_1^{\theta} + \nabla a^{\theta} A_1^{\theta} + a^{\theta} \nabla A_1^{\theta} - \frac{\mu}{\mu+1} \Delta (M_e \nabla A_1^{\theta}) - \nabla (g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}) - \nabla g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
P_0 \nabla g_2 &= \nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla (\nabla g_2) = \nabla (\nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla g_2) - \nabla g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \\
&= \varepsilon \nabla B_2^{\theta} - \Delta (M \nabla A_2^{\theta}) - \Delta (M_i \nabla A_1^{\theta}) - \nabla (g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}) - \nabla g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}.
\end{aligned}$$

Using the same method to preceding estimates and Lemma 3.3, it follows that

$$\begin{aligned}
&s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |\nabla g_1|^2 dx + s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |\nabla g_2|^2 dx \\
&\leq C(\gamma_1, \gamma_2, \gamma_3)(E_3 + E_4 + E_7) + C s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} (|\nabla g_1|^2 + |\nabla g_2|^2) dx. \tag{25}
\end{aligned}$$

Combing the above three estimates (23), (24) and (25), the proof is complete.  $\square$

In order to prove the main conclusion, we need to get further estimations for  $E_j$ ,  $j = 1, 2, 3, 4$ . The Carleman estimate in the previous section plays an important role in obtaining these estimations.

**Lemma 3.4.** *Assume all the conditions in Theorem 2.1 are satisfied. Then there exists a constant  $C$  depending only on  $\tilde{C}$ , such that for any  $\lambda \geq \lambda_0$  and  $s \geq s_1(\Omega, T)$ , the following inequality holds:*

$$E_1 + E_2 \leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2), \quad (26)$$

where  $\mathcal{E}(g_1, g_2, B_1, B_2)$  is defined as follows

$$\mathcal{E}(g_1, g_2, B_1, B_2) = \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx. \quad (27)$$

*Proof.* Note that  $\alpha(0, x) = +\infty$ . As a result of (11) and (12), we have

$$\begin{aligned} & \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|B_1^\theta|^2 + |B_2^\theta|^2) dx \\ &= \int_0^\theta \frac{\partial}{\partial t} \left( \int_\Omega \phi^{-1} e^{-2s\alpha} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dx \right) dt \\ &= \int_{Q_\theta} \phi^{-1} (-2s) \partial_t e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx - \int_{Q_\theta} \phi^{-2} \partial_t \phi e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \\ &\quad + 2 \int_{Q_\theta} \phi^{-1} e^{-2s\alpha} (2B_1 \partial_t B_1 + 2B_2 \partial_t B_2) dt dx \\ &\leq C(sT^5 + s\lambda T^8 + T^7) \int_Q \phi^3 e^{-2s\alpha} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dt dx \\ &\quad + (s\lambda)^{-1} \int_Q \phi^{-1} e^{-2s\alpha} (|\partial_t B_1|^2 + |\partial_t B_2|^2) dt dx \\ &\leq C(s\mathcal{I}(B_1) + \mathcal{I}(B_2)), \end{aligned}$$

where  $Q_\theta = (0, \theta) \times \Omega$ ,  $\mathcal{I}(B_j)|_{j=1,2}$  is defined in (14) and (16), for any  $s \geq C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)$  and  $\lambda \geq 1$ . Then using the estimate given in Theorem 2.1 to the system (18), we obtain

$$\begin{aligned} & s\mathcal{I}(B_1) + \mathcal{I}(B_2) \\ &\leq \tilde{C} \left( \int_Q e^{-2s\alpha} (s|F'|^2 + |G'|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + \int_Q s e^{-2s\alpha} |\partial_t a(t, x)| \left( \int_0^t |B_1(s, x)| ds \right. \right. \\ &\quad \left. \left. + |A_1(0, x)| \right)^2 dt dx \right) \\ &\leq \tilde{C} \left( s \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + C_1 T s \int_Q e^{-2s\alpha} |B_1|^2 dt dx \right). \end{aligned} \quad (28)$$

Due to this term  $C_1 T \int_Q e^{-2s\alpha} |B_1|^2 dt dx$  can be absorbed by  $\mathcal{I}(B_1)$ ,  $\lambda > 1$  and  $s$  being large enough, we have

$$\mathcal{I}(B_1) + \mathcal{I}(B_2) \leq \tilde{C}_1 s \lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2).$$

Thus for  $s \geq s_1 = \max\{s_0, C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)\}$ , the proof is complete.  $\square$

**Lemma 3.5.** *Let Assumption 3.1 be satisfied. Then there exists  $\lambda_1 = \max\{\lambda_0, C(\gamma_1, \gamma_2, \gamma_3)\}$  and  $s_2 = \max\{s_1, C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)\}$  for all  $\lambda \geq \lambda_1$ ,  $s \geq s_2$ , the following inequality holds:*

$$E_3 + E_4 \leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2),$$

where  $\mathcal{E}(g_1, g_2, B_1, B_2)$  is defined in (27).

*Proof.* First, we define

$$\pi(B_1) := e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1).$$

We multiply the first equation in (18) by  $\pi(B_1)$  and we integrate over  $Q_\theta$ , the result is

$$\int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) = \int_{Q_\theta} \pi(B_1) \left( \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F'(g_1, \nabla \tilde{v}^\varepsilon) \right) dt dx. \quad (29)$$

We divide (29) into left and right sides integrals to estimate separately. Firstly, we integrate the left side integral by parts, and get

$$\begin{aligned} & - \int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) dt dx \\ &= - \int_{Q_\theta} c_m \partial_t B_1 e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) dt dx \\ &= \int_{Q_\theta} c_m \partial_t B_1 \nabla(e^{-2s\alpha} \phi^{-1}) M_e \nabla B_1 dt dx + \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \quad (30)$$

Note that  $|\nabla(\phi^{-1} e^{-2s\alpha})| \leq s\lambda e^{-2s\alpha}$  for  $s \geq CT^2$ . Thus we have

$$\begin{aligned} \mathcal{J}_1 &\leq s\lambda(C\|M_e\|_{L^\infty(\Omega)}^2) s\lambda \int_{Q_\theta} \phi e^{-2s\alpha} |\nabla B_1|^2 dt dx + (s\lambda)^{-1} \int_{Q_\theta} \phi^{-1} e^{-2s\alpha} |\partial_t B_1|^2 dt dx \\ &\leq s\lambda \mathcal{I}(B_1) \end{aligned} \quad (31)$$

for any  $s \geq C(\gamma_2)T^2$ . Integrating by parts with respect to time in  $\mathcal{J}_2$ , we have

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= -\frac{1}{2} \int_{Q_\theta} c_m |\nabla B_1|^2 \partial_t (e^{-2s\alpha} \phi^{-1}) M_e dt dx \\ &\quad + \frac{1}{2} \int_{\Omega} (c_m |\nabla B_1|^2 e^{-2s\alpha} \phi^{-1} M_e) |_{t=\theta} dx. \end{aligned} \quad (32)$$

Here,

$$\begin{aligned} |\partial_t (e^{-2s\alpha} \phi^{-1})| &= |e^{-2s\alpha} \phi^{-2} \phi_t + e^{-2s\alpha} \phi^{-1} (-2s) \alpha_t| \\ &= |e^{-2s\alpha} \phi^{-1} (\phi^{-1} \phi_t - 2s \alpha_t)| \\ &\leq |e^{-2s\alpha} \phi^{-1}| \left( \frac{T^2}{4} + 2s \right) CT \phi^2 \\ &\leq Cs\lambda^2 \phi e^{-2s\alpha}, \end{aligned}$$

for  $\lambda > 1$  and  $s \geq CT^3$ . Therefore,

$$\mathcal{J}_2 \geq \frac{1}{2} \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2) |_{t=\theta} dx - Cs\lambda^2 \int_{Q_\theta} c_m \phi e^{-2s\alpha} M_e |\nabla B_1|^2 dt dx. \quad (33)$$

Now coming to the right side integrals of (29), we have

$$\begin{aligned} & \int_{Q_\theta} \pi(B_1) \left( \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F' \right) dt dx \\ &= \int_{Q_\theta} \pi(B_1) F' dt dx + \int_{Q_\theta} \pi(B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\ & \quad - \int_{Q_\theta} \pi(B_1) \partial_t a \left( \int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx - \int_{Q_\theta} \pi(B_1) a B_1 dt dx \\ &= \sum_{j=1}^4 \mathcal{K}_j. \end{aligned} \quad (34)$$

Then we estimate the above integrals one by one. Applying the Cauchy inequality, we get the following estimates for  $\mathcal{K}_{j=1,2}$ .

$$\begin{aligned} \mathcal{K}_1 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) F' dt dx \\ &= \int_{Q_\theta} e^{-s\alpha} \phi^{-\frac{1}{2}} F' (e^{-s\alpha} \phi^{-\frac{1}{2}} \nabla M_e \nabla B_1 + e^{-s\alpha} \phi^{-\frac{1}{2}} M_e \Delta B_1) dt dx \\ &\leq \int_{Q_\theta} CT^2 e^{-2s\alpha} |F'|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx \\ & \quad + \int_{Q_\theta} CT^4 \phi e^{-2s\alpha} |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\ &\leq s\lambda^2 (\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx), \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathcal{K}_2 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla(M_e \nabla B_1)|^2 dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla M_e \nabla B_1 + M_e \Delta B_1|^2 dt dx \\ &\leq \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} (2|\nabla M_e|^2 |\nabla B_1|^2 + 2|M_e|^2 |\Delta B_1|^2) dt dx \\ &\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\ &\leq s\lambda^2 \mathcal{I}(B_1), \end{aligned} \quad (36)$$

where  $\lambda \geq 1$  and  $s \geq C(\gamma_1)T^4$ . Next, we estimate the integral  $\mathcal{K}_3$ , and obtain

$$\mathcal{K}_3 = - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \partial_t a(t, x) \left( \int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx$$

$$\begin{aligned}
&\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e \Delta B_1 + \nabla M_e \nabla B_1|^2 dt dx + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
&\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\
&\quad + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
&\leq s\lambda^2 \mathcal{I}(B_1).
\end{aligned} \tag{37}$$

Similarly, we have

$$\mathcal{K}_4 = - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} a(t, x) B_1 \nabla (M_e \nabla B_1 + M_e \Delta B_1) dt dx \leq s\lambda^2 \mathcal{I}(B_1). \tag{38}$$

Using the assumptions on the conductivity  $M_e$  and substituting the inequalities (35)-(38) into (29), we get

$$-\mathcal{J}_1 - \mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j \leq s\lambda^2 \left( \mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right),$$

which means

$$-\mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j + \mathcal{J}_1 \leq s\lambda^2 \left( \mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right) + s\lambda \mathcal{I}(B_1).$$

Substituting (33) to the above inequality, we have

$$\begin{aligned}
&\left| \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2) |_{t=\theta} dx \right| \\
&\leq |\mathcal{J}_1| + \sum_{j=1}^4 \mathcal{K}_j \\
&\leq s\lambda^2 \left( \mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right) + s\lambda \mathcal{I}(B_1),
\end{aligned} \tag{39}$$

which leads to

$$\left| \int_{\Omega} e^{-2s\alpha} (\phi^\theta)^{-1} |\nabla B_1^\theta|^2 dx \right| \leq s\lambda^2 \left( \mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right). \tag{40}$$

Next we multiply the second equation of (18) by  $\xi(B_2) := e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2)$ , and integrate over  $Q_\theta$  to get

$$\begin{aligned}
&\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \varepsilon \partial_t B_2 dt dx \\
&= \int_{Q_\theta} \xi(B_2) \nabla(M \nabla B_2) dt dx + \int_{Q_\theta} \xi(B_2) \nabla(M_i \nabla B_1) dt dx + \int_{Q_\theta} \xi(B_2) G' dt dx \\
&= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla(M \nabla B_2)|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx \\
&\quad + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) G' dt dx.
\end{aligned}$$

We estimate

$$\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx$$

$$\begin{aligned}
&= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} (\nabla M \nabla B_2 + M \Delta B_2) \nabla (M_i \nabla B_1) dt dx \\
&\leq \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M \nabla B_2 + M \Delta B_2|^2 dt dx \\
&\quad + \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i \nabla B_1 + M_i \Delta B_1|^2 dt dx \\
&\leq CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M|^2 |\nabla B_2|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M|^2 |\Delta B_2|^2 dt dx \\
&\quad + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i|^2 |\nabla B_1|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_i|^2 |\Delta B_1|^2 dt dx \\
&\leq s\lambda^2 \mathcal{I}(B_1) + s\lambda^2 \mathcal{I}(B_2). \tag{41}
\end{aligned}$$

Continuing the similar computation as the preceding estimates and using Assumption 1.1, we obtain

$$\begin{aligned}
&| \int_{\Omega} e^{-2s\alpha^\theta} (\phi^\theta)^{-1} |\nabla B_2^\theta|^2 dx | \\
&\leq s\lambda^2 (\mathcal{I}(B_1) + \mathcal{I}(B_2)) + \int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx \\
&\leq Cs\lambda^2 \left( \int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \right), \tag{42}
\end{aligned}$$

for any  $s \geq C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)$  and  $\lambda \geq C(\gamma_1, \gamma_2, \gamma_3)$ . Thus combining the estimates (40) and (42), we obtain the conclusion.  $\square$

Now we shall give the main result of the stability estimate of the conductivities in (6) based on the preceding lemmas and proposition.

**Theorem 3.6.** *Let  $(A_1, A_2)$  be the solution of (7). Suppose all the assumptions of Theorem 2.1 hold and  $g_1, g_2 \in H_0^2(\Omega)$ . In addition, suppose Assumption 3.1 and 3.2 are also satisfied. Then there exists a constant  $C$  with  $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$ , such that for sufficiently large  $\lambda \geq \lambda_0 \geq 1$  and  $s \geq s_4$ , the following estimate holds:*

$$\begin{aligned}
&\int_{\Omega} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
&\leq C \left( \int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\
&\quad + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx. \tag{43}
\end{aligned}$$

*Proof.* Substituting the results in Lemma 3.4 and Lemma 3.5 into the inequality in Proposition 1, one obtains

$$\begin{aligned}
&s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
&\leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2) + C \sum_{j=5}^9 E_j(\theta) \\
&\leq Cs\lambda^2 \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + Cs^4 \lambda^6 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx
\end{aligned}$$

$$\begin{aligned}
& + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |\nabla A_1^{\theta}|^2 + |\Delta A_1^{\theta}|^2 + |A_1^{\theta}|^2 \right) dx + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |\nabla A_2^{\theta}|^2 + |\Delta A_2^{\theta}|^2 \right) dx \\
& + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}|^2 + |g_2 \Delta \tilde{u}_{e,\theta}^{\varepsilon}|^2 \right) dx + C \int_{\tilde{\omega}} \left( |g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\
& + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |A_1^{\theta}|^2 + \sum_{j=1}^2 \left( |\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) + |\nabla(g_1 \Delta \tilde{v}_{\theta}^{\varepsilon})|^2 \right. \\
& \left. + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^{\varepsilon})|^2 \right) dx \\
& \leq C s \lambda^2 \int_Q e^{-2s\alpha} \left( |F'|^2 + |G'|^2 \right) dt dx + C s^4 \lambda^6 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha} \left( |B_1|^2 + |B_2|^2 \right) dt dx \\
& + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |A_1^{\theta}|^2 + \sum_{j=1}^2 \left( |\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) \right) dx \\
& + C \int_{\tilde{\omega}} \left( |g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx,
\end{aligned}$$

for large enough  $s \geq s_3 = \max\{CT^2, s_2\}$  and  $\lambda \geq \lambda_1$ . Now for convenience, we set  $R_1(t, x) = \nabla \tilde{v}^{\varepsilon}(t, x)$  and  $R_2(t, x) = \nabla \tilde{u}_e^{\varepsilon}(t, x)$ . Then from the regularity of the solutions  $(\tilde{v}^{\varepsilon}(t, x), \tilde{u}_e^{\varepsilon}(t, x))$ , we deduce that there exist  $l_j \in L^2(0, T)$ ,  $j = 1, 2, 3, 4$ ,

$$|\partial_t R_j(t, x)| \leq l_j(t) |R_j^{\theta}|, \quad j = 1, 2,$$

$$|\partial_t \nabla R_1(t, x)| \leq l_3(t) |\nabla R_1^{\theta}|,$$

$$|\partial_t \nabla R_2(t, x)| \leq l_3(t) |\nabla R_2^{\theta}|,$$

for any  $(t, x) \in Q$ , and the functions  $l_j \in L^2(0, T)$ , implying  $\int_0^T |l_j|^2 dt \leq N < \infty$ ,  $j = 1, 2, 3, 4$ . Then we show

$$F' = \partial_t (\nabla(g_1 \nabla \tilde{v}^{\varepsilon})) = \nabla g_1 \partial_t R_1 + g_1 \partial_t \nabla R_1,$$

$$G' = \partial_t (\nabla(g_2 \nabla \tilde{u}_e^{\varepsilon})) = \nabla g_2 \partial_t R_2 + g_2 \partial_t \nabla R_2.$$

Observe that from the definition of  $\alpha$ , we get easily  $e^{-2s\alpha(t, x)} \leq e^{-2s\alpha^{\theta}}$  for all  $(t, x) \in Q$ . This implies

$$\begin{aligned}
& s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} \left( |g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\
& \leq C s^4 \lambda^6 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha} \left( |B_1|^2 + |B_2|^2 \right) dt dx + C \int_{\tilde{\omega}} \left( |g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx, \\
& + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left( |A_1^{\theta}|^2 + \sum_{j=1}^2 \left( |\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) \right) dx,
\end{aligned}$$

for sufficiently large  $s \geq s_4 = \max\{CT^2 N, s_3\}$ . From the properties of  $\alpha$  and  $\phi$ , there exist  $e_0$  and  $e_1$  such that

$$\inf_{x \in \Omega} \left( \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \right) \geq e_0 > 0,$$

$$\sup_{x \in \Omega} \left( \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \right) \leq e_1 < \infty.$$



Furthermore,  $e^{-\epsilon\alpha}\phi^m \leq C < \infty$  for all  $\epsilon > 0$  and  $m \in \mathbb{R}$  in  $Q_\omega$ . Thus we obtain

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\ & \leq C s^4 \lambda^6 \int_{Q_\omega} (|B_1|^2 + |B_2|^2) dt dx + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx, \\ & \quad + C \int_{\Omega} \left( |A_1^\theta|^2 + \sum_{j=1}^2 (|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|) \right) dx, \end{aligned}$$

Then we fix the parameters  $s, \lambda$  as  $s = s_4, \lambda = \lambda_1$ . This concludes the proof of the theorem.  $\square$

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